# THE GHOST LENGTH, LEVELS AND SHRIEK MAPS ON CLASSIFYING SPACES

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ABSTRACT. We give a lower bound of the cochain type level of the diagonal map on the classifying space of a Lie group by using the ghostness of a shriek map. Moreover, in a derived category, we discuss the triviality of the shriek map which induces the loop product on the classifying space of a Lie group.

## 1. INTRODUCTION

The *level* of an object in a triangulated category was introduced by Avramov, Buchweitz, Iyengar and Miller in [1]. The numerical invariant counts the number of steps to build the given object from some fixed object via triangles. On the other hand, Jørgensen [10, 11, 12] developed categorical representation theory of spaces employing the singular (co)chain complexes of spaces. Along the line of the work and by using the notion of the level, the author defined the *cochain type level* of a continuous map and investigated its fundamental properties in [16]. Furthermore, variants of the level and relationship between them and other topological invariants, for example the L.-S. category, are considered in [17, 18]. Thus the computations of the levels are in our interests.

String topology initiated by the fascinating paper of Chas and Sullivan [4] describes a rich structure in the homology of the free loop space $LM$ of a closed orientable manifold $M$. A basic one of the string operations is the so-called loop product on the shifted homology $H_{*+\dim M}(M)$. The key to defining these operations is to construct a shriek map (an Umkehr map or a wrong way map) on the homology. Félix and Thomas [8] generalized the construction of shriek maps on manifolds to that on Gorenstein spaces. This enables us to develop string topology in appropriate derived categories; see [20] for torsion functor descriptions of loop (co)products and their applications. It is important to mention that the class of Gorenstein spaces contains the classifying spaces of simply-connected Lie groups, Borel constructions more general, Poincaré duality spaces and hence closed oriented manifolds; see [6, 21].

Let $BG$ be the classifying space of a connected Lie group $G$. In [5], Chataur and Menichi showed that the homology $H_*(LBG; \mathbb{K})$ with coefficients in a field $\mathbb{K}$ carries the structure of homological conformal field theory (HCFT). The integration along the fibre of a Borel fibration plays a crucial role in defining the HCFT operations. We observe that in the setting the integration is considered a shriek map; see [8, Theorems 5 and 13]. In particular case, the shriek map which gives rise to the loop (co)products on $BG$ is a morphism of differential graded modules (henceforth called

DG modules) over the singular cochain algebra $C^*(BG; \mathbb{K})$. Thus one might take an interest in behavior of the shriek map in the derived category $\mathrm{D}(C^*(BG; \mathbb{K}))$ of DG modules over the DG algebra $C^*(BG; \mathbb{K})$.

In this short paper, we give a lower bound of the cochain type level of the diagonal map $\Delta : BG \to BG \times BG$ on the classifying space of $G$. Moreover, we show that the shriek map which gives rise to the loop product on $BG$ is trivial in the derived category $\mathrm{D}(C^*(BG; \mathbb{K}))$. Both the results indeed are extracted from a nice working relationship between shriek maps and the cochain type levels of maps. We stress that an intermediary in the interaction is the *ghost length* introduced by Hovey and Lockridge in [9]. An argument on the ghostness of the shriek map is the heart of this paper.

The rest of the paper is organized as follows. In Section 2, we describe our results after recalling the definition the level of a map. The goal of Section 3 is to proving the results.

## 2. Results

We recall the definition of the level. Let $A$ be a DG algebra over a field and $\mathrm{D}(A)$ the derived category of DG $A$-modules, namely the localization of the homotopy category $\mathrm{H}(A)$ of DG $A$-modules with respect to quasi-isomorphisms; see [13]. Observe that $\mathrm{D}(A)$ is a triangulated category with the shift functor $\Sigma$ defined by $(\Sigma M)^n = M^{n+1}$ and that a triangle in $\mathrm{D}(A)$ comes from a cofibre sequence of the form $M \xrightarrow{f} N \to C_f \to \Sigma M$ in the homotopy category $\mathrm{H}(A)$. Here $C_f$ denotes the mapping cone of $f$. We denote by $\mathtt{add}^\Sigma(A)$ the smallest strict full subcategory of $\mathrm{D}(A)$ that contains $A$ and is closed under finite direct sums and all shifts.

The category $\mathtt{smd}(A)$ is defined to be the smallest full subcategory of $\mathrm{D}(A)$ that contains $A$ and is closed under retracts. For full subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{D}(A)$, let $\mathcal{A} * \mathcal{B}$ be the full subcategory whose objects $L$ occur in a triangle $M \to L \to N \to \Sigma M$ with $M \in \mathcal{A}$ and $N \in \mathcal{B}$. We define $n$th thickening $\mathtt{thick}^n_{\mathrm{D}}(A)$ by

$$\mathtt{thick}^n_{\mathrm{D}(A)}(A) = \mathtt{smd}((\mathtt{add}^\Sigma(A))^{*n}),$$

where $\mathtt{thick}^0_{\mathrm{D}}(A) = \{0\}$; see [1, 2.2.1] and [3]. We then define a numerical invariant $\mathrm{level}_{\mathrm{D}(A)}(M)$ for an object $M$ in $\mathrm{D}(A)$, which is called the *level of $M$*, by

$$\mathrm{level}_{\mathrm{D}(A)}(M) := \inf\{n \in \mathbb{N} \cup \{0\} \mid M \in \mathtt{thick}^n_{\mathrm{D}(A)}(A)\}.$$

If no such integer exists, we set $\mathrm{level}_{\mathrm{D}(A)}(M) = \infty$.

Throughout what follows, $C^*(X)$ denotes the singular cochain algebra of a space $X$ with coefficients in a field $\mathbb{K}$ of arbitrary characteristic. Let $f : s(f) \to B$ be a map in the category of topological spaces. Then the cochain type level of the map $f$ is defined to be the level of $C^*(s(f))$ in the triangulated category $\mathrm{D}(C^*(B))$, namely $\mathrm{level}_{\mathrm{D}(C^*(B))}(C^*(s(f)))$.

Let $BG$ be the classifying space of a connected Lie group $G$. Since the diagonal map $\Delta : G \to G \times G$ is a homomorphism, it induces a map $BG \to BG^{\times 2}$, which is regarded as the diagonal map $BG \to BG \times BG$ under a homotopy equivalence between $BG^{\times 2}$ and $BG \times BG$. We give an estimate for the cochain type level of the composite

$$\Delta^{(n-1)} : BG \xrightarrow{B\Delta} BG^{\times 2} \longrightarrow \cdots \xrightarrow{B(1 \times \Delta)} BG^{\times n}$$

by considering the ghostness of the shriek maps of $B(1 \times \Delta) : BG^l \to BG^{\times(l+1)}$; see Section 3 for the definition of a ghost map. The result is described as follows.

**Theorem 2.1.** *Let $BG$ be the classifying space of a connected Lie group $G$ whose cohomology with coefficients in $\mathbb{K}$ is isomorphic to a polynomial algebra. Then*

$$n \leq \mathrm{level}_{\mathrm{D}(C^*(BG^{\times n}))}(C^*(BG)) \leq (n-1) \dim QH^*(BG; \mathbb{K}) + 1,$$

*where $QH^*(BG; \mathbb{K})$ stands for the vector space of indecomposable elements of the algebra $H^*(BG; \mathbb{K})$. Assume further that $QH^*(BG; \mathbb{K})^{2j+1} = 0$ for $j \geq 0$. Then*

$$\mathrm{level}_{\mathrm{D}(C^*(BG^{\times n}))}(C^*(BG)) = (n-1) \dim QH^*(BG; \mathbb{K}) + 1.$$

*Remark* 2.2. Let $F_{\Delta^{(n-1)}}$ be the homotopy fibre of the map $\Delta^{(n-1)} : BG \to BG^{\times n}$. Then the fibration $F_{\Delta^{(n-1)}} \to BG$ admits the holonomy right action of $\Omega BG^{\times n}$ and is weakly equivalent to the fibration $BG \times_{BG^{\times n}} EG^{\times n} \to BG$ with the holonomy right action of $G^{\times n}$; see [7, Proposition 2.11] for example. Then the duality property described in [18, Theorem 1.4(2)] deduces that

$$\mathrm{level}_{\mathrm{D}(C^*(BG^{\times n}))}(C^*(BG)) = \mathrm{level}^{\mathbb{K}}_{\mathrm{D}(C_*(G^{\times n}))}(C_*(BG \times_{BG^{\times n}} EG^{\times n})),$$

where the right-hand side denotes the *chain type* $\mathbb{K}$-*level* of the right $C_*(G^{\times n})$-module $C_*(BG \times_{BG^{\times n}} EG^{\times n})$; see [18, Introduction] for more details. Then under the same assumption as in Theorem 2.1, we need $n$ steps at least to built $C_*(BG \times_{BG^{\times n}} EG^{\times n})$ from the trivial module $\mathbb{K}$ via triangles in the triangulated category $\mathrm{D}(C_*(G^{\times n}))$. We observe that $BG \times_{BG^{\times n}} EG^{\times n}$ is homotopy equivalence to $G^{\times n}/\Delta G = G^{\times(n-1)}$, where $\Delta : G \to G^{\times n}$ is the diagonal map.

Let $M$ be a $\mathbb{K}$-Gorenstein space of dimension $d$; that is, $M$ is a simply-connected space which satisfies the condition that

$$\dim \mathrm{Ext}^m_{C^*(M)}(\mathbb{K}, C^*(M)) = \left\{ \begin{array}{ll} 0 & \text{if } m \neq d, \\ 1 & \text{if } m = d. \end{array} \right.$$

Then the dual to the loop product $Dlp : H^*(LM) \to H^{*+d}(LM \times LM)$ is defined by employing the shriek map $q^!$ of the inclusion $q : LM \times_M LM \to LM \times LM$. In order to recall the definition of $Dlp$ precisely, we consider a diagram

$$
\begin{array}{ccccc}
LM & \xleftarrow{\ Comp\ } & LM \times_M LM & \xrightarrow{\ q\ } & LM \times LM \\
{\scriptstyle ev_0} \downarrow & & {\scriptstyle ev_0} \downarrow & & \downarrow {\scriptstyle ev_0 \times ev_0} \\
M & =\!=\!=\!=\!= & M & \xrightarrow[\ \Delta\ ]{} & M \times M,
\end{array}
$$

where $Comp$ is the map induced by the product of loops on $M$. The result [8, Theorem 1] enables us to obtain the shriek map $q^! : C^*(LM \times_M LM) \to C^{*+d}(LM \times LM)$ of $q$. In fact $q^!$ is an extension of an appropriate shriek map $\Delta^! : C^*(M) \to C^{*+d}(M \times M)$ of the diagonal map; see the proof of [8, Theorem 2]. Then the dual to the loop product $Dlp : H^*(LM) \to (H^*(LM) \otimes H^*(LM))^{*+d}$ is defined to be the map on the cohomology induced by the composite

$$q^! \circ (Comp)^* : C^*(LM) \to C^*(LM \times_M LM) \to C^{*+d}(LM \times LM).$$

of morphisms of $C^*(M)$-modules. Let $G$ be a connected Lie group. We observe that $BG$ is a $\mathbb{K}$-Gorenstein space of dimension $-\dim G$ for any field $\mathbb{K}$ if $G$ is simply-connected or $H^*(BG; \mathbb{K})$ is a polynomial algebra; see [6, 21].

One of the main results in [19] asserts that $Dlp$ is trivial if $H^*(BG)$ is a polynomial algebra; see also [8, Theorem D] and [23, Theorem 4.7] for the case where the

characteristic of the underlying field is odd or zero. In fact, the map $H^*(q^!)$ is trivial; see the proof of [19, Proposition 3.1]. In general, we call a map $f : M \to N$ in the derived category $\mathrm{D}(A)$ of DG modules over a DG algebra $A$ a *ghost* if $H(f) = 0$. Thus the map $q^!$ is a ghost in $\mathrm{D}(C^*(BG))$. The following result states that $q^!$ and $Dlp$ are essentially trivial.

**Theorem 2.3.** *Suppose that* $H^*(BG)$ *is a polynomial algebra. Then the ghost map* $q^!$ *is trivial in* $\mathrm{D}(C^*(BG))$.

The shriek map which defines the loop coproduct on $BG$ is not trivial in general. In fact the loop coproduct itself is non-trivial even in the case where $H^*(BG)$ is a polynomial algebra; see [19, Theorem 1.1].

## 3. Proofs of Theorems 2.1 and 2.3

We recall here a numerical invariant for DG modules related to the level. Let $A$ be a DG algebra. An object $M$ in $\mathrm{D}(A)$ is said to have *ghost length* $n$, denoted gh.len.$M = n$, if every composite

$$M \xrightarrow{f_1} Y_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} Y_{n+1}$$

of $n + 1$ ghosts is trivial in $\mathrm{D}(A)$, and there exists a composite of $n$ ghosts from $M$ is non trivial in $\mathrm{D}(A)$; see [9].

The ghost length of a DG module $M$ gives a lower bound of the level of $M$.

**Proposition 3.1.** [22, Lemma 6.7] [17, Proposition 7.5] *For any* $M \in \mathrm{D}(A)$, *one has*

$$\mathrm{gh.len.}M + 1 \leq \mathrm{level}_{\mathrm{D}(A)}(M).$$

**Lemma 3.2.** $n - 1 \leq \mathrm{gh.len.}C^*(BG)$.

*Proof of Theorem 2.1.* We have a fibration of the form $G^{\times(n-1)} \to BG \xrightarrow{\Delta^{(n)}} BG^{\times n}$. Therefore we see that $H^*(G^{\times(n-1)}) \cong \mathrm{Tor}_*^{H^*(BG^{\times n})}(H^*(BG), \mathbb{K})$ and hence the torsion product is of finite dimension. Then it follows from [17, Lemma 7.1] that $\mathrm{level}_{\mathrm{D}(C^*(BG^{\times n}))}(C^*(BG)) \leq \mathrm{pd}_{H^*(BG^{\times n})}H^*(BG) + 1$, where $\mathrm{pd}_A M$ denotes the projective dimension of an $A$-module $M$.

Let $\mathcal{K} = H^*(BG) \otimes \wedge(s^{-1}QH^*(BG)) \otimes H^*(BG) \to H^*(BG)$ be the two sided bar resolution of $H^*(BG)$; see [2] for example. Then we have a projective resolution

$$\mathcal{K}^{\otimes_{H^*(BG)}^{n-1}} \to H^*(BG)^{\otimes_{H^*(BG)}^{n-1}} = H^*(BG)$$

of $H^*(BG)$ as an $H^*(BG)^{\otimes n-1}$-module. This yields that

$$\mathrm{pd}_{H^*(BG^{\times n})}H^*(BG) \leq (n-1)\dim QH^*(BG).$$

We have the upper bound. Proposition 3.1 and Lemma 3.2 give the lower bound.

We prove the latter half of the assertion. Since $H^*(BG; \mathbb{K})$ is a polynomial algebra generated by elements with even degree, it follows from [16, Proposition 2.4] that the homotopy fibration $G^{\times(n-1)} \to BG \xrightarrow{\Delta^{(n)}} BG^{\times n}$ is $\mathbb{K}$-formalizable; see[17, Section 2]. Thus the result [17, Proposition 5.2] implies that

$$\mathrm{level}_{\mathrm{D}(C^*(BG^{\times n}))}(C^*(BG)) = (n-1)\dim QH^*(BG; \mathbb{K}) + 1.$$

This completes the proof.                                                                                   $\square$

*Proof of Lemma 3.2.* By assumption $H^*(BG)$ is a polynomial algebra, say $H^*(BG) = \mathbb{K}[x_1, ..., x_s]$. Then $H^*(G)$ is isomorphic to the algebra with a 2-simple system of generators $s^{-1}x_1, ..., s^{-1}x_s$, where $\deg s^{-1}x_i = \deg x_i - 1$. Observe that $H^*(G)$ is the exterior algebra generated by $s^{-1}x_1, ..., s^{-1}x_s$ if the characteristic of $\mathbb{K}$ is odd.

**Claim 3.3.** *In the Leray-Serre spectral sequence $\{{}_{LS}E_r^{*,*}, d_r\}$ of the fibration $G \to BG^{k-1} \overset{B(1 \times \Delta)}{\to} BG^k$, the generators $s^{-1}x_i$ are transgressive. More precisely, for the transgression $\tau$, $\tau(s^{-1}x_i) = \lambda_i(x_i \otimes 1 - 1 \otimes x_i)$ for some non-zero scalar $\lambda_i$ under an isomorphism $H^*(BG^k) \cong H^*(BG^{k-2}) \otimes H^*(BG) \otimes H^*(BG)$.*

Therefore there is no non-trivial element in ${}_{LS}E_\infty^{0,*}$ for $* > 0$. This implies that that the shriek map $B(1 \times \Delta)^! : C^*(BG^{k-1}) \to C^{*-d}(BG^k)$ is a ghost map, where $d = \dim G$. In fact the induced map $H^*(B(1 \times \Delta)^!)$ is the integration along the fibre; see [8, Theorems 5 and 13].

We shall prove that the composition of the shriek maps $B(1 \times \Delta)^! \circ \cdots \circ B\Delta^! : C^*(BG) \to C^{*-(n-1)d}(BG^{n-1})$ is non-trivial in $\mathrm{D}(C^*(BG^{\times n}))$. To this end, letting $LBG$ be the space of free loops on $BG$, consider the homotopy pull-back square

$$
\begin{array}{ccccc}
G^{n-1} & \longrightarrow & LBG \times_{BG} \cdots \times_{BG} LBG & \overset{\widetilde{\Delta}}{\longrightarrow} & LBG \\
\| & & {\scriptstyle ev_0}\downarrow & & \downarrow{\scriptstyle ev_{a_1,...,a_n}} \\
G^{n-1} & \longrightarrow & BG & \underset{\Delta^{(n-1)}}{\longrightarrow} & BG^{\times n}
\end{array}
$$

with the homotopy fibration $\Delta^{(n-1)}$, where $ev_{a_1,...,a_n}$ denotes the evaluation map at points $a_k = \frac{k-1}{n}$ for $k = 1, ...., n$. We regard the composite $B(1 \times \Delta)^! \circ \cdots \circ B\Delta^!$ as the shriek map $(\Delta^{(n-1)})^!$ by choosing an appropriate orientation class of the fibration $\Delta^{(n-1)}$; see [5, Section 2.3, Composition] for example. In order to show non-triviality of the shriek map $(\Delta^{(n)})^!$, it suffices to prove that the shriek map $\widetilde{\Delta}^! : C^*(LBG \times_{BG} \cdots \times_{BG} LBG) \to C^{*-(n-1)d}(LBG)$ is non-trivial since $\widetilde{\Delta}^!$ is an extension of $(\Delta^{(n)})^!$; see the proof of [8, Theorem 6]. We observe that $H^*(LBG) \cong H^*(BG) \otimes \Delta(s^{-1}x_1, ...., s^{-1}x_s)$ as an algebra.

Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence for the pull-back above. Computing the $E_2$-term by using the projective resolution $\mathcal{K}^{\otimes_{H^*(BG)}^{n-1}} \to H^*(BG)$ mentioned in the proof of the upper bounds in Theorem 2.1, we see that

$$
\begin{aligned}
E_2^{*,*} &\cong H\big(H^*(BG)^{\otimes n} \otimes \Delta(s^{-1}x_{1,1}, ...., s^{-1}x_{1,s}, ...., s^{-1}x_{n-1,1}, ...., s^{-1}x_{n-1,s}) \\
&\quad \otimes_{H^*(BG)^{\otimes n}} H^*(LBG), \ \partial(s^{-1}x_{i,j}) = ev_{a_1,...a_n}^*(x_{i,j} \otimes 1 - 1 \otimes x_{i+1,j})\big)
\end{aligned}
$$

as a bigraded algebra. Since $ev_0 \simeq ev_{a_k}$ for any $k$, it follows that $ev_{a_1,...a_n}^* \circ p_k^* \simeq ev_0$, where $p_k : BG^{\times n}$ denotes the projection into the $k$-th factor. This implies that $ev_{a_1,...a_n}^*(x_{i,j} \otimes 1 - 1 \otimes x_{i+1,j}) = 0$ since $ev_0^*(x_i) = x_i$. For dimensional reasons, we see that $E_\infty^{*,*} \cong H^*(BG)^{\otimes n} \otimes \Delta(s^{-1}x_{1,1}, ...., s^{-1}x_{1,s}, ...., s^{-1}x_{n-1,1}, ...., s^{-1}x_{n-1,s})$. This fact enables us to conclude that the Leray-Serre spectral sequence of the upper fibtation in the homotopy pull-back above collapses at the $E_2$-term. Therefore it follows that the integration along the fibre $H^*((\widetilde{\Delta})^!)$ is non-trivial. We have the result. $\square$

*Proof of Claim 3.3.* We consider a morphism of homotopy fibrations

$$
\begin{array}{ccc}
G & =\!=\!=\!=\!=\!= & G \\
\downarrow & & \downarrow \\
BG^{k-2} \times BG & \xleftarrow{\ \simeq\ } & BG^{k-1} \\
{\scriptstyle 1 \times \Delta_{BG}}\downarrow & & \downarrow{\scriptstyle B(1\times\Delta)} \\
BG^{k-2} \times BG \times BG & \xleftarrow[\ \simeq\ ]{} & BG^{k}
\end{array}
$$

in which horizontal maps are homotopy equivalences. Thus in order to prove Claim 3.3, it suffices to show the assertion for the spectral sequence of the fibration $G \to BG \overset{\Delta_{BG}}{\to} BG \times BG$.

Let $z_i : BG \to K := K(\mathbb{K}, \deg x_i)$ be the map corresponding to the generator $x_i$ of $H^*(BG)$; that is, $z_i^*(\iota) = x_i$ for the fundamental class $\iota$ of $K$. In the Leray-Serre spectral sequence of the homotopy fibration $K(\mathbb{K}, \deg x - 1) \to K \overset{\Delta_K}{\to} K \times K$, the transgression sends the fundamental class of the fibre to the element $\iota \otimes 1 - 1 \otimes \iota$ up to the multiplication by a non-zero scalar because $\Delta_K^*(\iota \otimes 1 - 1 \otimes \iota) = 0$. The naturality of the morphism induced by $z_i$ implies that $\tau(s^{-1}x_i) = \lambda_i(x_i \otimes 1 - 1 \otimes x_i)$ for some non-zero scalar $\lambda_i$. We have the result. $\qquad\square$

*Proof of Theorem 2.3.* By assumption $H^*(BG)$ is a polynomial algebra. Therefore, by virtue of [14, Theorem 1.2 and Remark 3.4], we see that $H^*(LBG)$ is a free $H^*(BG)$-module. Suppose that $q^!$ is non-trivial. Then it follows from Proposition 3.1 and [17, Lemma 7.1] that

$$
\begin{aligned}
1 + 1 \quad &\leq \quad \text{gh.len.}C^*(LBG) + 1 \leq \text{level}_{\mathrm{D}(C^*(BG))}(C^*(LBG)) \\
&\leq \quad \text{pd}_{H^*(BG)}H^*(LBG) + 1 = 0 + 1,
\end{aligned}
$$

which is a contradiction. This completes the proof. $\qquad\square$

We conclude this article with an important remark.

*Remark* 3.4. In the proof of Lemma 3.2, it is proved that the shriek map $\Delta^! : C^*(BG) \to C^{*-\dim G}(BG \times BG)$ is non-trivial ghost map if $H^*(BG)$ is a polynomial algebra. This follows from the fact that the extension $\widetilde{\Delta}^!$ of $\Delta^!$ is non-trivial. On the other hand, Theorem 2.3 asserts that another extension $q^!$ of $\Delta^!$ is trivial in $\mathrm{D}(C^*(BG))$. In consequence, all maps between derived functors given by the map $q^!$, for example maps on the torsion functor or on the extension functor which $q^!$ defines, are trivial.

## References

[1] L. L. Avramov, R. -O. Buchweitz, S. B. Iyengar and C. Miller, Homology of perfect complexes, Adv. Math. **223**(2010), 1731-1781. arXiv: `math.AC/0609008v2`.

[2] P. F. Baum and L. Smith, Real cohomology of differential Fibre bundles, Comment. Math. Helv. **42**(1967), 171-179.

[3] A. Bondal and M. Ban den Bergh, Generators and representability of functors in commutative and non-commutative geometry, Moscow Math. J. **3**(2003), 1-36.

[4] M. Chas and D. Sullivan, String topology, preprint (math.GT/0107187).

[5] D. Chataur and L. Menichi, String topology of classifying spaces, preprint (2008) arXiv:`math.AT/0801.0174`.

[6] Y. Félix, S. Halperin and J. -C. Thomas, Gorenstein spaces. Adv. in Math. **71**(1988), 92-112.

[7] Y. Félix, S. Halperin and J. -C. Thomas, Rational Homotopy Theory, Graduate Texts in Mathematics **205**, Springer-Verlag.

[8] Y. Félix and J. -C. Thomas, String topology on Gorenstein spaces, Math. Ann. **345**(2009), 417-452.

[9] M. Hovey and K. Lockridge, The ghost dimension of a ring, Proc. Amer. Math. Soc., **137** (2009), 1907-1913.

[10] P. Jørgensen, Auslander-Reiten theory over topological spaces, Comment. Math. Helv. **79**(2004), 160-182.

[11] P. Jørgensen, The Auslander-Reiten quiver of a Poincaré duality space, Algebr. Represent. Theory **9**(2006), 323-336.

[12] P. Jørgensen, Calabi-Yau categories and Poincaré duality spaces, preprint (2008), arXiv:`math.RT/0801.2052v2`.

[13] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) **27**(1994), 63-102.

[14] A. Kono and K. Kuribayashi, Module derivations and cohomological splitting of adjoint bundles, Fundamenta Mathematicae, **180**(2003), 199-221.

[15] I. Kriz and J. P. May, Operads, algebras, modules and motives. Astérisque, no. 233,1995.

[16] K. Kuribayashi, On the levels of maps and topological realization of objects in a triangulated category, J. Pure and Appl. Algebra, **216**(2012) 752-765, arXiv: `math.AT/1102.3271`.

[17] K. Kuribayashi, Upper and lower bounds of the (co)chain type level of a space, to appear in Algebra and Representation Theory, arXiv:`math.AT/1006.2669`.

[18] K. Kuribayashi, Duality on the (co)chain type levels, preprint (2011), arXiv: `math.AT/1108.3890v2`.

[19] K. Kuribayashi and L. Menichi, On the loop (co)products on the classifying space of a Lie group, in preparation.

[20] K. Kuribayashi, L. Menichi and T. Naito, Torsion functor description of loop (co)products, in preparation.

[21] A. Murillo, The virtual Spivak fiber, duality on fibrations and Gorenstein spaces, Trans. Amer. Math. Soc. **359** (2007), 3577-3587.

[22] K. Schmidt, Auslander-Reiten theory for simply connected differential graded algebras, preprint (2008), arXiv:`math.RT/0801.0651v1`.

[23] H. Tamanoi, Stable string operations are trivial, Int. Math. Res. Not. IMRN 2009, **24**, 4642-4685.